General reaction-diffusion processes with separable equations for correlation functions

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(Received 21 April 2002; revised manuscript received 25 July 2002; published 29 October 2002)

We consider general multispecies models of reaction diffusion processes and obtain a set of constraints on the rates which give rise to closed systems of equations for correlation functions. Our results are valid in any dimension and on any type of lattice. We also show that under these conditions the evolution equations for two point functions at different times are also closed. As an example we introduce a class of two species models that may be useful for the description of voting processes or the spreading of epidemics.

I. INTRODUCTION

An interesting class of nonequilibrium problems with a rich dynamical behavior and a vast area for applications are stochastic reaction-diffusion systems (see [1–4] and references therein). These are the processes in which one or several species of particles hop randomly on a lattice, and interact in various possible ways with each other. In the one species case, it is convenient to denote a particle by the symbol 1 and a vacant site (a hole) by the symbol 0. Then a simple hopping is represented by $1+0 \to 0+1$. In addition to exclusion which means that no two particles can occupy a single site, the possible interactions include pair creation $(0+0 \to 1+1)$, pair annihilation $(1+1 \to 0+0)$, coagulation $(1+1 \to 1+0)$, decoagulation, $(0+1 \to 1+1)$, birth $(0 \to 1)$ and death $(1 \to 0)$ processes. Obviously the variety of elementary processes rapidly increases with the number of species.

In general, such lattice systems are difficult to treat by rigorous analytical means and correspondingly, considering the vast amount of such models, relatively few exact results are known.

Over the past few years the application of operator formalism to these stochastic processes and their mapping to quantum spin systems and their generalizations has turned to be quite fruitful. In view of this correspondence, many of the techniques of quantum spin systems such as free fermion techniques, Bethe ansatz and related algebraic techniques [5–11], have been used in the investigation of reaction-diffusion systems, specially in one-dimensional lattice systems. Also by using the operator formalism, some of the old techniques such as the matrix product ansatz [12] have been put to very fruitful use, in solution of one-dimensional stochastic systems [13] (see Refs. [1,2,14] and references therein). Almost all of the above methods have one limitation, they are restricted to one-dimensional lattices, specially if we are interested in exact solutions.

A common feature of any model of interacting particles and indeed the main source of difficulty in obtaining exact solutions is that the equations of motion of correlation functions form an infinite hierarchy, that is, the equation of motion of n-point functions include $(n+1)$-point functions and

in general higher correlation functions. One can truncate the hierarchy at a level by various kinds of approximations, the simplest and the most common method is to break the hierarchy at the first level by the mean field approximation. However, in some models and in low space dimensions, the amount of diffusive mixing may not be enough to warrant such an approximation. Some of these models may also have long relaxation times so that their simulation may be difficult and time consuming. For these models exact solutions are highly desirable.

Interestingly enough, there are models in which the hierarchy of equations of correlation functions automatically breaks at every level, hence the possibility of obtaining exact solutions. For these models the equations of motion of n-point functions depend only on k-point functions with $k \le n$. We should stress that while a powerful technique like matrix product ansatz or its dynamical version [15] only transform the problem to another equally difficult and yet convenient problem, that is, finding the representations and calculating matrix elements of strings of operators of an algebra [13,16,17], for these models the simplification appears to be genuinely effective. Moreover this property is independent of the dimension and the geometry of the lattice. In view of its generality this is a great simplification and deserves to be pursued further from various directions.

The observation of this phenomenon in some models like symmetric exclusion and partial exclusion processes [4,18–20] led G. M. Schütz [21] to raise the question of classification of such models, i.e., a general criterion on the reaction and diffusion rates such that the resulting equations for the correlation functions decouple. By considering the one species processes and the particle density correlation functions $\langle n(x) \rangle$, he found that from among the 12-parameter family of single species reaction diffusion systems, a 10-parameter family fall within this class.

Since then this question has been pursued further. For example, one can consider the hole-density correlation functions $\langle 1-n(x) \rangle$ (the so called empty interval method) with two [23,24] or three site interactions [25] or even more general functions like $\langle a+bn(x) \rangle$ [22,23,26]. These correlation functions lead to different sets of constraints on the rates.

The aim and the results of the paper

The aim of the present paper is to investigate this question for the general multispecies case. That is, we assume that

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there are $p+1$ species of particles labeled as type $0,1,2,\dots p$ which hop and interact in a lattice of arbitrary geometry. We interpret particles of type 0 as holes and other particles as real particles. We assume two body interactions between the particles, and obtain the general condition on the rates, so that the correlation functions of densities of real particles, i.e., particles of type $1,2,\dots p$ decouple from the correlation functions of higher levels in the hierarchy. We show that under this condition the equation of motion of two-time two-point correlation functions are also closed. These types of correlation functions are important in the analysis of voting processes [27].

In view of the arbitrary number of species in our model, one will have much more freedom to find an exactly solvable model for description of physical phenomena, specially in the area of chemical kinetics, where one usually has more than one species of particles.

Since in our analysis and hence in our results, there is an asymmetry between a particle of type 0 and other particles, it is important to keep in mind that the interpretation of particle 0 as a hole is not essential. Therefore in adapting a model from the class discussed below to a situation of physical interest, one can change this interpretation and take any other particle of type $1,2,\dots p$ to stand for a hole and particle 0 for a real particle.

For general p-species models, the number of independent rates is $(p+1)^4-(p+1)^2$. We show that the family of exactly solvable models (in the above sense) live on an N_p-dimensional manifold (hyperplane) where $N_p:=(p+1)^4-(p+1)^2-2p^3$. For the one-species case we find $N_1=16-4-2=10$, in accordance with the results of Schütz [21]. For the two- and three-species cases we have, respectively, $N_2=56$ and $N_3=186$, where the number of independent rates for these models are originally 72 and 240, respectively. For an elaboration on this see the remark 3 in the text.

As an example and for concreteness we study a two-species family and further constrain it with certain extra symmetries. In this class we can find models suitable for the description of the spread of an epidemic, the exchange of ideas and votes, and the spreading of news or rumor.

We set up the solution of one-point functions for these models and investigate to a certain extent the properties of these solutions.

The structure of this paper is as follows: In Sec. II we introduce our notations and conventions. In Sec. III we obtain the general conditions on the rates. In Sec. IV we consider the case $p=2$, and by imposing further symmetry requirement on these models we introduce a class of solvable two-species models. In Sec. V we set up the general solution of the one point functions for this model. We conclude the paper with a discussion.

II. NOTATIONS AND CONVENTIONS

Throughout the paper we will adhere to notations which we will collect here for convenience. *For ease of notations we will consider a one-dimensional lattice which may be infinite or periodic. However, all our results are valid also on lattices of arbitrary shape and arbitrary dimension. We will*

also work explicitly with nearest neighbor interactions, although again our results are valid for arbitrary range of interactions. These facts have already been shown in Ref. [21] and are also easily verified by reviewing our method of proof, in the sense that no part of the reasoning depends on the underlying lattice or the range of interactions.

We denote the points of the lattice by Latin letters from the end of the alphabet $x-1,x,x+1,\dots$. In a finite lattice we number the sites from 1 to L. To each site x of the lattice, we assign a random variable $\tau(x)$ which can take $p+1$ values $0,1,\dots p$. We denote the values of this random variable when it takes all the values including possibly the value 0, by Greek letters $\alpha,\beta,\mu,\dots$, and when it takes only the values different from 0, by Latin letters, from the middle of the alphabet, like $i,j,k,l\cdot$. Thus $\langle n_i(x)\rangle:=\langle\delta_{\tau(x),i}\rangle$ denotes the average density of particles of type i at site x, or the probability of site x being occupied by a particle of type i, and $\langle n_0(x)\rangle:=\langle\delta_{\tau(x),0}\rangle$ denotes the probability of this site being empty. It thus follows that

$$\sum_{\mu=0}^{p}\langle n_\mu(x)\rangle=1. \tag{1}$$

We assume that the two particles of type α and β on two adjacent sites may transform stochastically to particles of type μ and ν with rate $R_{\alpha,\beta}^{\mu,\nu}$. This is written as

$$\alpha,\beta\rightarrow\mu,\nu \quad\text{with rate}\quad R_{\alpha\beta}^{\mu\nu}. \tag{2}$$

Since a Greek index includes also the value 0, which we interpret as a vacant site, the above transformations include all the possible processes conceivable for all types of particles on the two sites. For example R_{i0}^{0i} is the hopping rate of a particle of type i and R_{00}^{k0} is the rate of creation of a particle of type k from the vacuum and R_{ij}^{k0} is the rate with which two particles of type i and j interact or coagulate to form a particle of type k.

We will use the operator formalism for Markov processes. This formalism is well known by now, since in the past few years it has been used extensively for the analysis of reaction-diffusion processes, specially in one dimension.

In this formalism we should assign a complex $(p+1)$-dimensional Hilbert space C^{p+1} to each site, with orthonormal basis states

$$|0\rangle,|1\rangle,\dots|p\rangle, \quad \langle\mu|\nu\rangle=\delta_{\mu,\nu}. \tag{3}$$

The Hilbert space of the whole lattice is the tensor product of all the local Hilbert spaces of the sites. At any given time t, each configuration of the lattice is given by the values of the random variables of all sites $\tau(1),\tau(2),\dots\tau(L)$. Such a configuration occurs with probability $P(\tau(1),\tau(2),\dots\tau(L);t)=\langle\tau(1),\tau(2),\dots\tau(L)|P(t)\rangle$. The state vector $|P(t)\rangle$ determines all the probabilities and its evolution is governed by a Hamiltonian derived from the rates:

$$\frac{d}{dt}|P\rangle=H|P\rangle. \tag{4}$$

For convenience we have absorbed the negative sign, which is usually included in this equation, into the definition of the Hamiltonian. The probabilities are normalized by requiring that

$$\sum_{\tau(1),\tau(2),\ldots\tau(L)} \langle \tau(1),\tau(2),\ldots\tau(L)|P\rangle=1. \qquad (5)$$

This equation can be rewritten as

$$\langle S|P\rangle:=\langle s|^{\otimes L}|P\rangle=1, \qquad (6)$$

where $\langle s|$ is defined as

$$\langle s|:=\sum_{\mu=0}^{p}\langle\mu|. \qquad (7)$$

Note that the Bra state $\langle S|$ is the sum of all the possible configurations of the system.

From the master equation (4) and the property $\langle S|H=0$ (as required by conservation of probability), one obtains the Heisenberg-like equation of motion for the average of any time independent observable O:

$$\frac{d}{dt}\langle O(t)\rangle=\langle S|[\hat{O},H]|P(t)\rangle, \qquad (8)$$

where $\hat{O}$ is an appropriately chosen operator whose matrix element gives the average of the observable, i.e.,

$$\langle O(t)\rangle=\langle S|\hat{O}|P(t)\rangle. \qquad (9)$$

Note that the time dependence of the average comes from the evolution of probabilities, therefore a better notation for the average of a time independent observable will be $\langle O\rangle(t)$. However, we use the notation $\langle O(t)\rangle$, since it will be convenient when we consider two point functions at unequal times. Consider a completely general Markov system whose configurations are labeled by C and a time independent observable O of the configurations. We have

$$\langle O(t_2)O(t_1)\rangle:=\sum_{C_2,C_1}O(C_2)O(C_1)P(C_2,t_2;C_1,t_1). \qquad (10)$$

In the operator formalism, it is easily shown that this two-point function is represented as

$$\langle O(t_2)O(t_1)\rangle:=\langle S|\hat{O}(t_2)\hat{O}(t_1)|P(0)\rangle, \qquad (11)$$

where $\hat{O}(t)$ is the Heisenberg-like operator, $\hat{O}(t):=e^{-tH}\hat{O}e^{tH}$. The above correspondence is also true for different kinds of operators. In particular we have, $\langle O(t)O(0)\rangle:=\langle S|e^{-tH}\hat{O}e^{tH}\hat{O}|P(0)\rangle$. We then have

$$\frac{d}{dt}\langle O(t)O(0)\rangle=\langle S|[\hat{O}(t),H]\hat{O}|P(0)\rangle$$

$$=\langle S|[\hat{O},H]e^{tH}\hat{O}|P(0)\rangle, \qquad (12)$$

where we have used $\langle S|e^{-tH}=\langle S|$.

Returning to our model, the local operators $E_{\alpha,\beta}:=|\alpha\rangle\langle\beta|$ act on the states of a site as: $E_{\alpha,\beta}|\mu\rangle=\delta_{\beta,\mu}|\alpha\rangle$. Of particular interest are the diagonal operators $E_{00},E_{11},\ldots E_{pp}$ which act as number operators for the holes, particles of type 1, to particles of type p. Hereafter we abbreviate these diagonal operators and show them only by one index instead of two, i.e., E_μ stands for $E_{\mu\mu}$:

$$E_0=\begin{pmatrix}1 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0\end{pmatrix}\cdots E_p=\begin{pmatrix}0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 1\end{pmatrix}. \qquad (13)$$

The above operators have the commutation relations

$$[E_{\alpha\beta},E_{\mu\nu}]=\delta_{\beta\mu}E_{\alpha\nu}-\delta_{\nu\alpha}E_{\mu\beta}. \qquad (14)$$

Moreover the following property of these operators is also important in the sequel:

$$\langle s|E_{\alpha\beta}=\langle s|E_{\beta\beta}=:\langle s|E_\beta, \qquad (15)$$

where we have abbreviated $E_{\beta\beta}$ as E_β.

The Hamiltonian describing the processes (2) is constructed as a sum of local Hamiltonians acting on adjacent sites

$$H=\sum_x h(x,x+1), \qquad (16)$$

where the operator $h(x,x+1)$ means that the operator h acts only nontrivially on sites x and $x+1$. The operator h is constructed from local operators as follows, where we will use hereafter the Einstein summation convention over Greek indices

$$h=R_{\alpha\beta}^{\mu\nu}(E_{\mu\alpha}\otimes E_{\nu\beta}-E_\alpha\otimes E_\beta). \qquad (17)$$

The conservation of probability constrains the rates to satisfy the relation

$$\sum_{\mu,\nu}R_{\alpha,\beta}^{\mu,\nu}=0 \quad \forall\alpha,\beta. \qquad (18)$$

Finally we need two matrices constructed from the matrix of rates which will prove useful later, and we prefer to introduce them here for convenience. Out of the matrix of rates, we form a set of matrices R^i and S^i defined as follows:

$$(R^i)_{\alpha,\beta}:=\sum_\nu R_{\alpha\beta}^{i\nu}, \qquad (19)$$

$$(S^i)_{\alpha,\beta}:=\sum_\nu R_{\alpha\beta}^{\nu i}. \qquad (20)$$

The conditions for decoupling of equations will be expressed in terms of these matrices.

III. DECOUPLING OF CORRELATION FUNCTIONS

The simplest correlation functions are one-point functions which determine the average densities of particles of each type at each site. Thus we are interested in the equation of motion of the one-point function $\langle n_i(x) \rangle := \langle \delta_{\tau(x),i} \rangle$. This is in fact the probability that site x is occupied with a particle of type i. In the operator formalism this one-point function is written as a matrix element of the corresponding operator, namely,

$$\langle n_i(x,t) \rangle = \langle S|E_i(x)|P(t)\rangle =: \langle E_i(x) \rangle, \quad (21)$$

where by the last equality we have defined the bracket or the average of an operator. From Eq. (8), we find

$$\frac{d}{dt}\langle n_i(x) \rangle = \langle [E_i(x),H] \rangle$$

$$= \langle [E_i(x),h(x-1,x)] \rangle + \langle [E_i(x),h(x,x+1)] \rangle. \quad (22)$$

Each of these two terms leads in general to two-point functions on their relevant sites, the first term on $(x-1,x)$ and the second on $(x,x+1)$. We now ask under what condition a cancellation occurs in *each* of these terms *separately* so that we end up with only one-point functions on the right hand site. A little reflection on the words written in italic shows that the question we are asking and henceforth its answer, does not depend at all on the underlying lattice and the range of the interaction, in so far as we are only considering Hamiltonians with two-body interactions. To find the answer to our question we calculate one of these terms say the second one. We note that

$$\langle S|[E_i(x),h(x,x+1)] = R^{\mu,\nu}_{\alpha\beta}\langle S|[E_{ii}(x),E_{\mu\alpha}(x)E_{\nu\beta}(x+1)], \quad (23)$$

where we have ignored the diagonal part of $h(x,x+1)$ which obviously commute with $E_{ii}(x)$. Using the commutation relations (14) and also the property (15) we find

$$\langle S|[E_{ii}(x),h(x,x+1)] = \sum_{\nu} R^{i\nu}_{\alpha\beta}\langle S|E_\alpha(x)E_\beta(x+1)$$

$$- \sum_{\mu,\nu} R^{\mu\nu}_{i\beta}\langle S|E_i(x)E_\beta(x+1). \quad (24)$$

The second term vanishes in view of Eq. (18) and we are left with

$$\langle S|[E_{ii},h(x,x+1)] = (\mathbf{R}^i)_{\alpha,\beta}\langle S|E_\alpha(x)E_\beta(x+1). \quad (25)$$

Expanding the right hand terms we find

$$\langle S|[E_{ii},h(x,x+1)] = \mathbf{R}^i_{0,0}\langle S|E_0(x)E_0(x+1)$$

$$+ \mathbf{R}^i_{j,0}\langle S|E_j(x)E_0(x+1)$$

$$+ \mathbf{R}^i_{0,k}\langle S|E_0(x)E_k(x+1)$$

$$+ \mathbf{R}^i_{j,k}\langle S|E_j(x)E_k(x+1), \quad (26)$$

where now we are using the summation convention on Latin indices and $\mathbf{R}^i_{j,k}$ stands for $(\mathbf{R}^i)_{j,k}$. We now use the operator identity

$$E_0 = 1 - E_1 - E_2 - \cdots E_p, \quad (27)$$

and eliminate the operators $E_0(x)$ and $E_0(x+1)$ on the right hand side of the above equation and demand that all the quadratic terms vanish. It is easily seen that this cancellation occurs when the matrices $\mathbf{R}^i$ satisfy the relations,

$$\mathbf{R}^i_{j,k} = \mathbf{R}^i_{0,k} + \mathbf{R}^i_{j,0} - \mathbf{R}^i_{0,0} \quad \forall i,j,k. \quad (28)$$

This simply means that for each matrix $\mathbf{R}^i$, all the elements are fixed once the elements of the first row and column are determined.

Similar calculations for the first term of Eq. (22) leads to the following condition:

$$\mathbf{S}^i_{j,k} = \mathbf{S}^i_{0,k} + \mathbf{S}^i_{j,0} - \mathbf{S}^i_{0,0} \quad \forall i,j,k, \quad (29)$$

where the matrices $\mathbf{S}^i$ have already been defined in Eq. (19).

Multiplying both sides of Eq. (26) by $|P(t)\rangle$ it is seen that once the above conditions are satisfied, the equations of motion of 1-point functions depend only on one point functions. A short calculation shows that once the equations of motion for one point functions are closed, it guarantees that the equation of higher-order correlation functions are also closed, that is their equation of motion depend solely on the k-point functions with $k \leq n$. In this way the hierarchy of equations of n-point functions is terminated and closed at any level and the system amends itself to exact analytical treatment. Every model whose rates satisfy the relations (28) and (29) is solvable in the above sense.

Moreover we show that the equations of motion of two-point functions at different times are also closed under the above conditions. In some models like voting models [27] the calculation of such correlation functions are important.

To see this consider the two-point *two time* correlation function $\langle n_i(x,t)n_j(x',0)\rangle$. We have

$$\frac{d}{dt}\langle n_i(x,t)n_j(x',0)\rangle \equiv \langle S|e^{-tH}[E_{ii}(x),H]e^{tH}E_{jj}(x')|P(0)\rangle \quad (30)$$

$$\equiv \langle S|[E_{ii}(x),H]e^{tH}E_{jj}(x')|P(0)\rangle. \quad (31)$$

Under the conditions (28) and (29), we know that the commutator in the above equation is expressible as the sum of local site operators. Restoring the operator e^{-tH}, i.e., $\langle S| \rightarrow \langle S|e^{-tH}$, we find that on the right hand side of this equa-

tion only two-point functions appear. Therefore, the equation of motion of two-time two-point functions will also be closed.

A. Remarks

(1) The emergence of two sets of constraint for the rates does have nothing to do with a given site x, having two neighbors $x-1$ and $x+1$ and hence on the underlying lattice. On a general lattice we still have only these two equations. It only reflects the fact that the interaction Hamiltonians defined on each link may not be symmetric under the interchange of its ends. Therefore each site x contributes to two types of interaction Hamiltonians, namely, those of the type $h(x,y)$ which for all different y's lead to Eq. (28) and those of the type $h(z,x)$ which for all different z's lead to Eq. (29). In fact for symmetric models in which there is no driving force one expects that these two sets of constraint become identical. This is in fact the case, since for these symmetric models, one has $R^{\mu\nu}_{\alpha\beta} = R^{\nu\mu}_{\beta\alpha}$ and hence $\mathbf{R}^i_{\alpha,\beta} = \mathbf{S}^i_{\beta,\alpha}$, which makes the two set of conditions identical.

(2) For general models the number of independent rates is $(p+1)^4 - (p+1)^2$. To count the number of conditions on the rates we note that a matrix such as $\mathbf{R}^1$ imposes p^2 linear equations on the rates, since its first row and column determine all the other elements. There are $2p$ such matrices and hence the number of conditions is $2p^3$. Therefore the number of parameters of the solvable family we are considering is $N_p = (p+1)^4 - (p+1)^2 - 2p^3$. For the one species case we find $N_1 = 16 - 4 - 2 = 10$. For the two- and three-species models we have 56 and 186 free parameters, respectively.

(3) It appears that the family of integrable multispecies models live on a manifold of huge dimension and one is at complete ease to choose many models of his own choice for adapting them to any physical situation. However, this is an illusion and as we will see, in choosing physically interesting models from this manifold, one is much more restricted than it appears at first sight. The reason is that we are usually interested in simple subclasses of these models, ones in which we can set many of the irrelevant or uninteresting parameters equal to zero either on physical grounds or to make our analysis simple and transparent. However, it often happens that once we set some of these parameters equal to zero, the set of admissible parameters collapses drastically so that we are left with totally uninteresting models or models which are actually equivalent to one-species models. Geometrically the constraints are a set of hyperplanes which pass through the origin. The admissible rates lie in the intersection of all these hyperplanes. However, all the rates are also constrained to be positive. It may happen that this intersection of hyperplanes, although a manifold of high dimension, may intersect the positive sector of the space of parameters in a very low-dimensional submanifold. As an example consider a model with say six rates r_1, r_2, r_3, r_4, r_5, and r_6 and one constraint $r_1 + r_2 + r_3 + r_4 + r_5 - r_6 = 0$. The set of admissible rates is obviously a five-dimensional plane. However, if for some reason we are interested in those models in which the rate r_6 is vanishing, then we are left with the constraint $r_1 + r_2 + r_3 + r_4 + r_5 = 0$ that forces all these remaining param-

eters to vanish. In the multispecies case this difficulty shows up more severely, since we are dealing with a large number of hyperplanes. Thus it is nontrivial to find physically interesting multispecies models in the class discussed above.

We will conclude this section with the final form of the equation of motion of one-point functions. Collecting the remaining linear terms in Eq. (26) and its counterpart, we find (with summation convention understood for Latin indices)

$$\frac{d}{dt}\langle n_i(x)\rangle = \mathbf{R}^i_{0,0}\langle n_0(x) + n_0(x+1) - 1\rangle + \mathbf{R}^i_{j,0}\langle n_j(x)\rangle$$
$$+ \mathbf{R}^i_{0,j}\langle n_j(x+1)\rangle + (x \to x-1, \mathbf{R} \to \mathbf{S}). \quad (32)$$

This equation is specific to a one-dimensional lattice. In a general lattice it should be modified appropriately, the modification is, however, straightforward (see Sec. IV). Before going to the consideration of a two-species model, it is instructive to recapitulate the findings for the one-species model from this general perspective.

B. The one-species separable models

If there is only one species of particles on the lattice, we have only two matrices, namely, $\mathbf{R}^1$ and $\mathbf{S}^1$. These are two by two matrices, subject to the conditions:

$$\mathbf{R}^1_{1,1} + \mathbf{R}^1_{0,0} = \mathbf{R}^1_{1,0} + \mathbf{R}^1_{0,1}, \quad (33)$$

$$\mathbf{S}^1_{1,1} + \mathbf{S}^1_{0,0} = \mathbf{S}^1_{1,0} + \mathbf{S}^1_{0,1}. \quad (34)$$

When expanded by using the definition (28) and (29) of the matrices $\mathbf{R}^1$ and $\mathbf{S}^1$, they yield:

$$R^{10}_{11} + R^{11}_{11} + R^{10}_{00} + R^{11}_{00} = R^{10}_{01} + R^{11}_{01} + R^{10}_{10} + R^{11}_{10}, \quad (35)$$

$$R^{01}_{11} + R^{11}_{11} + R^{01}_{00} + R^{11}_{00} = R^{01}_{01} + R^{11}_{01} + R^{01}_{10} + R^{11}_{10}. \quad (36)$$

One can now eliminate the diagonal terms from normalization to transform the above relations to

$$R^{00}_{10} + R^{01}_{10} + R^{10}_{00} + R^{11}_{00} = R^{10}_{01} + R^{11}_{01} + R^{00}_{11} + R^{01}_{11}, \quad (37)$$

$$R^{00}_{01} + R^{10}_{01} + R^{01}_{00} + R^{11}_{00} = R^{00}_{11} + R^{10}_{11} + R^{01}_{10} + R^{11}_{10}, \quad (38)$$

which are the relations given in Ref. [21]. In the following section we will consider a two-species model.

IV. A TWO-SPECIES SEPARABLE MODEL

As mentioned above the rates of the solvable two-species models of the type considered in this paper live on a 56-dimensional hyperplane. By exploring various regions of this plane, one can find interesting two-species models suitable for various applications. In this section we explore a small region of this plane by imposing extra symmetry requirements on the model.

We consider systems in which there is no driving force, i.e., systems which have the symmetry $R^{\mu,\nu}_{\alpha,\beta} = R^{\nu,\mu}_{\beta,\alpha}$. As noted above [see Eqs. (28) and (29)], for these models the **R** matrices and the **S** matrices lead to identical constraints. Furthermore, we restrict ourselves to those models in which an individual species is neither created nor annihilated but only

changes its label. Such models may be appropriate for description of voting processes or the spreading of epidemics. This means that we are setting $R_{0,0}^{i,0} = R_{i,0}^{0,0} = R_{i,0}^{k,0} = R_{i,0}^{j,k} = 0$. In other words the number of Latin indices should be equal as subscripts and superscripts of R. A rate such as $R_{1,0}^{2,0}$ means that a voter with vote 1 spontaneously (or due to the effect of environment) changes his or her vote to vote 2 (or a healthy individual 1 gets infected due to the interaction with the environment). In a time interval dt two voters with different votes 1 and 2, pass each other, without changing their votes, with probability $R_{12}^{21}dt$. It may also happen that on this close contact the voter 1 changes his or her idea and switch to vote 2. This will happen with probability $R_{12}^{22}dt$. Note that the voting processes that have been studied in the literature contain only two species $+$ and $-$ with no vacant site. Here we have also vacant sites, and the voters can move in free space and interact with each other. Equations (28) now yields:

$$R_{10}^{10} + R_{01}^{10} = R_{11}^{11} + R_{11}^{12}, \quad R_{10}^{20} + R_{01}^{20} = R_{11}^{21} + R_{11}^{22}, \quad (39)$$

$$R_{10}^{10} + R_{02}^{10} = R_{12}^{11} + R_{12}^{12}, \quad R_{10}^{20} + R_{02}^{20} = R_{12}^{21} + R_{12}^{22}, \quad (40)$$

$$R_{20}^{10} + R_{01}^{10} = R_{21}^{11} + R_{21}^{12}, \quad R_{20}^{20} + R_{01}^{20} = R_{21}^{21} + R_{21}^{22}, \quad (41)$$

$$R_{20}^{10} + R_{02}^{10} = R_{22}^{11} + R_{22}^{12}, \quad R_{20}^{20} + R_{02}^{20} = R_{22}^{21} + R_{22}^{22}. \quad (42)$$

The diagonal terms like $R_{10}^{10}, R_{20}^{20} \cdots$ are a source of trouble, since they are minus the sum of a number of rates and we would better get rid of them. To do so we proceed as follows: We note that in each pair of equations above, the sum of the right hand side terms adds up to zero, due to normalization. Thus the sum of the left hand sides must also add up to zero. If we do so and substitute for the diagonal terms their value from the normalization [i.e., substitute R_{10}^{10} with $-(R_{10}^{01} + R_{10}^{02} + R_{10}^{20})$], we find that the first and the last pair of equations of the above set lead to trivial identities. We also find that the second and the third pairs lead to one single identity, meaning that in each pair one is redundant. Thus we keep this last identity and safely ignore all the equations which contain diagonal terms. Therefore, we are actually dealing with five independent equations relating positive rates, the final forms of which are

$$R_{10}^{20} + R_{10}^{02} = R_{11}^{21} + R_{11}^{22}, \quad R_{20}^{10} + R_{20}^{01} = R_{22}^{12} + R_{22}^{11}, \quad (43)$$

$$R_{10}^{20} + R_{02}^{20} = R_{12}^{21} + R_{12}^{22}, \quad R_{20}^{10} + R_{01}^{10} = R_{21}^{12} + R_{21}^{11}, \quad (44)$$

$$R_{01}^{10} + R_{01}^{20} = R_{02}^{20} + R_{02}^{10}. \quad (45)$$

These are the final conditions on the rates for this kind of two-species model.

Equations of motion

To obtain the equations of motion for the above two species model, we use Eq. (32) and obtain for a d-dimensional rectangular lattice with unit vectors $\mathbf{e}_r$; $r = 1, \ldots d$, and with the abbreviation $\langle n_i(\mathbf{x}) \rangle \to n_i(\mathbf{x})$:

$$\frac{d}{dt} n_1(\mathbf{x}) = R_{01}^{10} \nabla^2 n_1(\mathbf{x}) - 2d(R_{01}^{02} + R_{01}^{20}) n_1(\mathbf{x}) + R_{02}^{10} \nabla^2 n_2(\mathbf{x})$$

$$+ 2d(R_{02}^{01} + R_{02}^{10}) n_2(\mathbf{x}), \quad (46)$$

and

$$\frac{d}{dt} n_2(\mathbf{x}) = R_{02}^{20} \nabla^2 n_2(\mathbf{x}) - 2d(R_{02}^{01} + R_{02}^{10}) n_2(\mathbf{x}) + R_{01}^{20} \nabla^2 n_1(\mathbf{x})$$

$$+ 2d(R_{01}^{02} + R_{01}^{20}) n_1(\mathbf{x}), \quad (47)$$

where $\nabla^2 n(\mathbf{x}) := (\sum_{r=1}^{r=d} n(\mathbf{x} + \mathbf{e}_r) + n(\mathbf{x} - \mathbf{e}_r) - 2dn(\mathbf{x}))$ stands for the discrete d-dimensional Laplacian.

All the terms in these equations can be understood intuitively. For example, a term like $R_{02}^{10} \nabla^2 n_2(\mathbf{x})$ measures the diffusion of particles of type 2 which change their type or color to 1 as they hop.

If one begins to write the equations intuitively taking all the complex interactions into account, one finds many other terms. But at the end they all cancel out. For the description of the solution, it is convenient to define new parameters:

$$\gamma_1 := 2d(R_{01}^{20} + R_{01}^{02}), \quad \gamma_2 = 2d(R_{02}^{10} + R_{02}^{01}), \quad (48)$$

$$\gamma := \gamma_1 + \gamma_2, \quad (49)$$

$$D := R_{01}^{10} + R_{01}^{20} = R_{02}^{20} + R_{02}^{10} \equiv 1, \quad (50)$$

$$D' := R_{01}^{10} - R_{02}^{10} = R_{02}^{20} - R_{01}^{20}, \quad (51)$$

where we have used the last relation of Eq. (43) in the last two relations and have rescaled time to set $D = 1$. These parameters have obvious physical interpretations, γ_1 and γ_2, respectively, determine the overall tendency of particles of type 1 and 2 to switch their types. D is the diffusion constant of the particles when we ignore their types and finally D' a kind of relative diffusion constant. It measures the difference of diffusion constant for the particles that do not change their type in hopping, compared with those which do so.

Solving these coupled system of differential-difference equations will give the distribution of both types of particles in space and time. In principle it is possible to go to Fourier space and diagonalize the resulting matrix equation. However, it is better to proceed in a more physically transparent way by defining new densities:

$$n(\mathbf{x}) = n_1(\mathbf{x}) + n_2(\mathbf{x}), \quad (52)$$

$$\phi(\mathbf{x}) = \gamma_1 n_1(\mathbf{x}) - \gamma_2 n_2(\mathbf{x}). \quad (53)$$

Here $n(\mathbf{x})$ is the total density of particles (when we ignore their types or colors) and $\phi(\mathbf{x})$ is a weighted difference of densities.

It is now a matter of simple algebra to use Eq. (48) to arrive at the following equations for these new densities:

$$\frac{\partial}{\partial t} n(\mathbf{x}) = \nabla^2 n(\mathbf{x}), \quad (54)$$

which means that if we ignore the color of particles, they perform simple diffusion with diffusion constant D. We also obtain

$$\frac{\partial}{\partial t}\phi(\mathbf{x}) = D'\nabla^2\phi(\mathbf{x}) + D''\nabla^2 n(\mathbf{x}) - \gamma\phi(\mathbf{x}), \quad (55)$$

where

$$D'' := R_{02}^{10}\gamma_1 - R_{01}^{20}\gamma_2 \equiv R_{02}^{10}R_{01}^{02} - R_{01}^{20}R_{02}^{01}. \quad (56)$$

The total number of particles of each species obey very simple equations which are obtained by summing the above equations over $\mathbf{x}$. Denoting the number of particles of species 1 and 2 by N_1 and N_2 respectively, and noting that $N_2 = N - N_1$, where N is the total number of particles, we find from Eq. (55) by summing ϕ over all the sites:

$$\frac{d}{dt}N_1 = -\gamma_1 N_1 + \gamma_2 N_2 = -\gamma_1 N_1 + \gamma_2(N - N_1), \quad (57)$$

the solution of which is

$$N_1(t) = \frac{\gamma_2}{\gamma_1+\gamma_2}N + e^{-(\gamma_1+\gamma_2)t}\left(N_1(0) - \frac{\gamma_2}{\gamma_1+\gamma_2}N\right), \quad (58)$$

$$N_2(t) = \frac{\gamma_1}{\gamma_1+\gamma_2}N + e^{-(\gamma_1+\gamma_2)t}\left(N_2(0) - \frac{\gamma_1}{\gamma_1+\gamma_2}N\right). \quad (59)$$

We now discuss the solution of Eqs. (54) and (55) which determine the spatial distribution of particles in time.

Let us define the Fourier transforms

$$\bar{n}(\mathbf{q},t) := \sum_{\mathbf{x}} e^{i\mathbf{q}\cdot\mathbf{x}}n(\mathbf{x}), \quad \bar{\phi}(\mathbf{q},t) := \sum_{\mathbf{x}} e^{i\mathbf{q}\cdot\mathbf{x}}\phi(\mathbf{x}), \quad (60)$$

where $\mathbf{q} = (q_1, q_2, \ldots q_d)$ and for all i, $q_i \in [0, 2\pi)$, from which we find:

$$n(\mathbf{x}) = \int_0^{2\pi} e^{-i\mathbf{q}\cdot\mathbf{x}}\bar{n}(\mathbf{q},t)\frac{d\mathbf{q}}{(2\pi)^d}$$

$$\phi(\mathbf{x}) = \int_0^{2\pi} e^{-i\mathbf{q}\cdot\mathbf{x}}\bar{\phi}(\mathbf{q},t)\frac{d\mathbf{q}}{(2\pi)^d}. \quad (61)$$

With the definition

$$Q = 2\sum_{r=1}^{r=d} (\cos q_r - 1), \quad (62)$$

we find the equations of motion for these generating functions as

$$\dot{\bar{n}} = Q\bar{n}, \quad (63)$$

$$\dot{\bar{\phi}} = Q(D'\bar{\phi} + D''\bar{n}) - \gamma\bar{\phi}, \quad (64)$$

with the general solution

$$\bar{n}(q,t) = \bar{n}(q,0)e^{Qt}, \quad (65)$$

$$\bar{\phi}(q,t) = e^{(QD'-\gamma)t}\left[\bar{\phi}(q,0) - \frac{QD''}{Q(1-D')+\gamma}\bar{n}(q,0)\right]$$

$$+ e^{Qt}\left[\frac{QD''}{Q(1-D')+\gamma}\bar{n}(q,0)\right]. \quad (66)$$

Once the initial distributions of particles of each type is known, these equations allow us to determine the distributions of both types of particles in later times. If the particles are in a finite volume, then the above solutions are still valid, except that the momenta $\mathbf{q}$ will take discrete values.

The large scale behavior of these densities is determined by going to the limit of $q \to 0$, where $Q \to -|\mathbf{q}|^2$. For illustration we consider two simple examples.

Example 1. Let us assume that the particles (voters) change their type (votes) only on encounter with other particles (voters). In this case we have: $R_{10}^{20} = R_{10}^{02} = R_{20}^{10} = R_{20}^{01} = 0$. From Eq. (43) we find that the only nonzero parameters which remain are $R_{12}^{21} =: P, R_{12}^{22} = R_{12}^{11} =: A$ and $R_{10}^{01} = R_{20}^{02} =: D$, subject to a relation: $D = P + A$, where we have introduced new simple labels for the rates. The labels D, P, and A stand, respectively, for "diffusion," "pass" and "agreement":

$$1\ 0 \leftrightarrow 0\ 1 \quad \text{with rate} \quad D, \quad (67)$$

$$2\ 0 \leftrightarrow 0\ 2 \quad \text{with rate} \quad D, \quad (68)$$

$$1\ 2 \leftrightarrow 2\ 1 \quad \text{with rate} \quad P, \quad (69)$$

$$1\ 2 \to 2\ 2 \quad \text{with rate} \quad A, \quad (70)$$

$$1\ 2 \to 1\ 1 \quad \text{with rate} \quad A. \quad (71)$$

In this case we find from Eqs. (46) and (47) that both types of particles diffuse through each other without any interaction

$$\frac{\partial}{\partial t}n_1(\mathbf{x}) = D\nabla^2 n_1(x), \quad \frac{\partial}{\partial t}n_2(\mathbf{x}) = D\nabla^2 n_2(x). \quad (72)$$

However, this is peculiar to one-point functions and the equations of two-point functions will indeed be coupled to each other by interaction parameters. This is an example of an observation first made in Ref. [32] according to which some Hamiltonians may lead to the same set of equations for one-point functions. Here our Hamiltonian is equivalent to a free Hamiltonian as far as the one-point functions are concerned.

Example 2. Let us now assume that one of the particles say type 1 does not change its type, or one kind of voters is persistent in his vote, that is, $R_{10}^{20} = R_{10}^{02} = 0$. The first equation of (43) then leads to $R_{11}^{21} = R_{11}^{22} = 0$. In this case we find from

Eq. (48) that $D''=0$ and $D'=1$. The relations (43) reduce to the following relations between the remaining rates of reactions:

$$R_{02}^{20}=R_{12}^{21}+R_{12}^{22}, \qquad (73)$$

$$R_{01}^{10}=R_{02}^{20}+R_{02}^{10}, \qquad (74)$$

$$R_{20}^{10}+R_{01}^{10}=R_{21}^{12}+R_{21}^{11}, \qquad (75)$$

$$R_{20}^{10}+R_{20}^{01}=R_{22}^{12}+R_{22}^{11}. \qquad (76)$$

For simplicity we first consider the large scale form of the distribution functions. From Eq. (65) we obtain

$$\bar{n}(\mathbf{q},t)=\bar{n}(\mathbf{q},0)e^{-|\mathbf{q}|^2 t}, \qquad (77)$$

$$\bar{\phi}(\mathbf{q},t)=\bar{\phi}(\mathbf{q},0)e^{(-|\mathbf{q}|^2-\gamma)t}. \qquad (78)$$

Let us consider a situation at time $t=0$ where there are N_2 particles of type 2 at the origin, i.e., $n_2(\mathbf{x},0)=N_2\delta(\mathbf{x})$ in a uniform set of particles of type 1 of density ρ, i.e., $n_1(\mathbf{x},0)=\rho$. Thus the initial values of Fourier transforms are

$$\bar{n}_1(\mathbf{q},0)=\rho(2\pi)^d\delta(\mathbf{q}) \quad \bar{n}_2(\mathbf{q},0)=N_2, \qquad (79)$$

or

$$\bar{n}(\mathbf{q},0)=N_2+\rho(2\pi)^d\delta(\mathbf{q}) \quad \bar{\phi}(\mathbf{q},0)=-\gamma_2 N_2. \qquad (80)$$

Inserting these into Eq. (77), we find

$$\bar{n}(q,t)=[N_2+\rho(2\pi)^d\delta(\mathbf{q})]e^{-|\mathbf{q}|^2 t}$$

$$\equiv N_2 e^{-|\mathbf{q}|^2 t}+\rho(2\pi)^d\delta(\mathbf{q}), \qquad (81)$$

$$\bar{\phi}(q,t)=-\gamma_2 N_2 e^{-(\gamma_2+|\mathbf{q}|^2)t}. \qquad (82)$$

Taking the inverse Fourier transform and using the definition (52) we find

$$n_1(\mathbf{x},t)=\rho+\frac{N_2}{(4\pi t)^{d/2}}e^{-|\mathbf{x}|^2/4t}(1-e^{-\gamma_2 t}), \qquad (83)$$

$$n_2(\mathbf{x},t)=\frac{N_2}{(4\pi t)^{d/2}}e^{-|\mathbf{x}|^2/4t}e^{-\gamma_2 t}. \qquad (84)$$

$$(85)$$

The initial number of particles of type 2 diffuse and gradually the particles of type 2 turn into type 1. At the end, no particle of type 2 remains.

We can also consider the small scale behavior of these distributions. In this case the solution of the Eqs. (63) is given by modified Bessel functions. We give below the solution for the initial distribution of one particle of type

2 at the origin and a uniform distribution of particles of type 1 at all other sites except the origin. That is $[n_2(\mathbf{x},0)=\delta_{\mathbf{x},0}$ or $\bar{n}_2(\mathbf{q},0)=1]$ and $[n_1(\mathbf{x})=\rho(1-\delta_{\mathbf{x},0})$ or, $(\bar{n}_1(\mathbf{q},0)=\rho((2\pi)^d\delta(\mathbf{q})-1]$. For these initial conditions we can find $\bar{n}(\bar{q},t)$ and $\bar{\phi}(\mathbf{q},t)$ from Eq. (65). The result is

$$\bar{n}(\mathbf{q},t)=(1-\rho)e^{Qt}+(2\pi)^d\rho\delta(\mathbf{q}), \qquad (86)$$

$$\bar{\phi}(\mathbf{q},t)=-\gamma_2 e^{-(Q-\gamma_2)t}. \qquad (87)$$

On the lattice we will find

$$n(\mathbf{x},t)=\rho+(1-\rho)\frac{1}{(2\pi)^d}\int_0^{2\pi}e^{Qt-i\mathbf{q}\cdot\mathbf{x}}d\mathbf{q}, \qquad (88)$$

$$\phi(\mathbf{x},t)=-\gamma_2 e^{-\gamma_2 t}\frac{1}{(2\pi)^d}\int_0^{2\pi}e^{Qt-i\mathbf{q}\cdot\mathbf{x}}d\mathbf{q}. \qquad (89)$$

These integrals can be written in terms of modified Bessel functions $I_x(t)=1/2\pi\int_0^{2\pi}e^{-iqx+\cos qt}dq$. Thus we find

$$n_1(\mathbf{x},t)=\rho+(1-\rho-e^{-\gamma_2 t})e^{-2dt}\prod_{\mathbf{x}}I_x(2t), \qquad (90)$$

$$n_2(\mathbf{x},t)=e^{-\gamma_2 t}e^{-2dt}\prod_{\mathbf{x}}I_x(2t). \qquad (91)$$

V. DISCUSSION

We have obtained the general condition under which the hierarchy of equations for n-point functions of a multispecies reaction-diffusion system truncates at all orders, that is the equations of motion of n-point functions depend only on those of lower correlation functions. Also under the above conditions the equations of motion of two-point functions at different times form a closed system. We have selected out a class of 2 species models which may be appropriate for the description of the spread of an epidemic or as a voter model. This work can be extended in several directions, even if one restricts oneself to the two-species models. First, one can study other two species models which include coagulation, decoagulation or birth and death processes. Second, in the two-species model one can change our interpretation of 0 particle as a hole and take it to be a real particle. In this way one can adapt the rates for description of other models. Third, one can study also the asymmetric models and relax our simplifying assumption on the symmetry of rates, and finally one can consider the similarity transformation [28–31] on the model to obtain the other exactly solvable models.

Note added in proof. After the original version of this paper was submitted I was informed by one of the authors of Ref. [32] that Eqs. (28) and (29) have already been obtained by these authors. Their considerations, however, are restricted to one dimensional lattices and to equal time correlation functions.

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